

Subordination and Superordination of the Liu-Srivastava Linear Operator on Meromorphic Functions

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Abstract. Using the methods of differential subordination and superordination, sufficient conditions are determined on the Liu-Srivastava linear operator of meromorphic functions in the punctured unit disk to obtain respectively the best dominant and the best subordinant. New sandwich-type results are also obtained.

2000 Mathematics Subject Classification: Primary 30C80; Secondary 30C45

Key words and phrases: Hypergeometric function, subordination, superordination, Liu-Srivastava linear operator, convolution, meromorphic function.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, with $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let f and F be members of $\mathcal{H}(U)$. The function f is said to be *subordinate* to F , or F is *superordinate* to f , if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$. Denote by \mathcal{Q} the set of all functions q that are analytic and injective on $\bar{U} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of \mathcal{Q} for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

The following classes of admissible functions will be required.

Definition 1.1. [17, Definition 2.3a, p. 27] Let Ω be a set in \mathbb{C} , $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those

functions $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\Re \left\{ \frac{t}{s} + 1 \right\} \geq k \Re \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular, if

$$q(z) = M \frac{Mz + a}{M + \bar{a}z}, \quad (M > 0, |a| < M),$$

then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}(a)$. In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.2. [18, Definition 3, p. 817] Let Ω be a set in \mathbb{C} , $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z)$, $s = zq'(z)/m$, and

$$\Re \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, Miller and Mocanu proved the following theorems.

Theorem 1.1. [17, Theorem 2.3b, p. 28] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ satisfies

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

Theorem 1.2. [18, Theorem 1, p. 818] Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U , then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}$$

implies $q(z) \prec p(z)$.

For a fixed $p \in \mathbb{N} := \{1, 2, \dots\}$, let Σ_p denote the class of all p -valent functions of the form

$$(1.1) \quad f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in U^* := \{z \in \mathbb{C} : 0 < |z| < 1\}).$$

For two functions f given by (1.1) and g given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \dots, l$) and $\beta_k \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($k = 1, 2, \dots, m$), the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_l)_k}{(\beta_1)_k \dots (\beta_m)_k} \frac{z^k}{k!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\}),$$

where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2) \dots (a+n-1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z^{-p} {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z),$$

the Liu-Srivastava operator [15, 16] $H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \Sigma_p \rightarrow \Sigma_p$ is defined by the Hadamard product

$$H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) := h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z)$$

$$= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p} \dots (\alpha_l)_{k+p}}{(\beta_1)_{k+p} \dots (\beta_m)_{k+p}} \frac{a_k z^k}{(k+p)!}.$$

For convenience, we write

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}_p(a, c) := H_p^{(2,1)}(1, a; c)$ (studied among others by Liu and Srivastava [13], Liu [14], and Yang [23]), the operator $D^{n+1} := \mathcal{L}_p(n+p, 1)$, which is analogous to the Ruscheweyh derivative operator (investigated by Yang [22]), and the operator

$$J_{c,p} := \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt = \mathcal{L}_p(c, c+1) \quad (c > 0)$$

(studied by Uralegaddi and Somanatha [21]). It is to be noted that the Liu-Srivastava operator investigated in [10, 19, 20] is the meromorphic analogue of the Dziok-Srivastava [9] linear operator.

Aghalary *et al.* [1, 2], Ali *et al.* [3, 4, 5, 6], Aouf and Hossen [8] and Kim and Srivastava [11] obtained sufficient conditions for certain differential subordination implications to hold. In particular, Liu and Owa [12] investigated a subordination problem for meromorphic functions defined through a linear operator D^n ; in fact, they have determined a class of admissible functions so that

$$\left| h \left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right) \right| < 1 \Rightarrow \left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1.$$

In the present investigation, by making use of the differential subordination and superordination results of Miller and Mocanu [17, Theorem 2.3b, p. 28] and [18, Theorem 1, p. 818], certain classes of admissible functions are determined so that subordination as well as superordination implications of functions associated with the Liu-Srivastava linear operator $H_p^{l,m}$ hold. Ali *et al.* [7] have considered a similar problem for the multiplier transformation on meromorphic functions. Additionally, several new differential sandwich-type results are obtained.

2. Subordination of the Liu-Srivastava linear operator

The following class of admissible functions is required in our first result.

Definition 2.1. Let Ω be a set in \mathbb{C} and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + \alpha_1 q(\zeta)}{\alpha_1}, \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1)$$

$$\Re \left\{ \frac{(\alpha_1 + 1)(w - u)}{v - u} - (2\alpha_1 + 1) \right\} \geq k \Re \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in U, \zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Theorem 2.1. Let $\phi \in \Phi_H[\Omega, q]$. If $f \in \Sigma_p$ satisfies

$$(2.1) \quad \left\{ \phi \left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z \right) : z \in U \right\} \subset \Omega,$$

then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. Define the analytic function p in U by

$$(2.2) \quad p(z) := z^p H_p^{l,m}[\alpha_1]f(z).$$

In view of the relation

$$(2.3) \quad \alpha_1 H_p^{l,m}[\alpha_1 + 1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1 + p)H_p^{l,m}[\alpha_1]f(z),$$

it follows from (2.2) that

$$(2.4) \quad z^p H_p^{l,m}[\alpha_1 + 1]f(z) = \frac{1}{\alpha_1} [\alpha_1 p(z) + zp'(z)].$$

Further computations show that

$$(2.5) \quad z^p H_p^{l,m}[\alpha_1 + 2]f(z) = \frac{1}{\alpha_1(\alpha_1 + 1)} \left[z^2 p''(z) + 2(\alpha_1 + 1)zp'(z) \right] + p(z).$$

Define the transformations from \mathbb{C}^3 to \mathbb{C} by

$$(2.6) \quad \begin{aligned} u(r, s, t) &= r, \\ v(r, s, t) &= \frac{\alpha_1 r + s}{\alpha_1}, \end{aligned}$$

$$w(r, s, t) = \frac{t + 2(\alpha_1 + 1)s + \alpha_1(\alpha_1 + 1)r}{\alpha_1(\alpha_1 + 1)}.$$

Let

$$\begin{aligned} \psi(r, s, t; z) &:= \phi(u, v, w; z) \\ (2.7) \quad &= \phi \left(r, \frac{\alpha_1 r + s}{\alpha_1}, \frac{t + 2(\alpha_1 + 1)s + \alpha_1(\alpha_1 + 1)r}{\alpha_1(\alpha_1 + 1)}; z \right). \end{aligned}$$

The proof will make use of Theorem 1.1. Using equations (2.2), (2.4) and (2.5), it follows from (2.7) that

$$(2.8) \quad \begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ &= \phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z). \end{aligned}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(\alpha_1 + 1)(w - u)}{v - u} - (2\alpha_1 + 1),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z). \quad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal map $h(z)$ of U onto Ω . In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_H[h, q]$ with $q(0) = 1$. If $f \in \Sigma_p$ satisfies*

$$(2.9) \quad \phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z) \prec h(z),$$

then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

Our next result is an extension of Theorem 2.1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U , $q(0) = 1$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ and*

$$\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z) \in \Omega,$$

then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $z^p H_p^{l,m}[\alpha_1]f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. \blacksquare

Theorem 2.3. *Let $h(z)$ and $q(z)$ be univalent in U , with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ satisfy one of the following conditions:*

- (1) $\phi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or
- (2) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ satisfies (2.9), then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. The result is similar to the proof of [17, Theorem 2.3d, p. 30] and is therefore omitted. ■

The next theorem yields the best dominant of the differential subordination (2.9).

Theorem 2.4. *Let $h(z)$ be univalent in U , and $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. Suppose the differential equation*

$$(2.10) \quad \phi \left(p(z), p(z) + \frac{zp'(z)}{\alpha_1}, \frac{z^2p''(z) + 2(\alpha_1 + 1)zp'(z)}{\alpha_1(\alpha_1 + 1)} + p(z); z \right) = h(z)$$

has a solution $q(z)$ with $q(0) = 1$, and satisfy one of the following conditions:

- (1) $q(z) \in \mathcal{Q}_1$ and $\phi \in \Phi_H[h, q]$
- (2) $q(z)$ is univalent in U and $\phi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or
- (3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ satisfies (2.9), then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Following the same arguments in [17, Theorem 2.3e, p. 31], we deduce that $q(z)$ is a dominant from Theorems 2.2 and 2.3. Since $q(z)$ satisfies (2.10), it is also a solution of (2.9) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. ■

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, can be expressed in the following form:

Definition 2.2. *Let Ω be a set in \mathbb{C} and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that*

$$(2.11) \quad \phi \left(1 + Me^{i\theta}, 1 + \frac{k + \alpha_1}{\alpha_1} Me^{i\theta}, 1 + \frac{L + (2k + \alpha_1)(1 + \alpha_1)Me^{i\theta}}{\alpha_1(1 + \alpha_1)}; z \right) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re(L e^{-i\theta}) \geq (k - 1)kM$ for all real θ , $\alpha_1 \in \mathbb{C}$ ($\alpha_1 \neq 0, -1$) and $k \geq 1$.

Corollary 2.2. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \Sigma_p$ satisfies*

$$\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z) \in \Omega,$$

then

$$|z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$. Corollary 2.2 can now be written in the following form:

Corollary 2.3. *Let $\phi \in \Phi_H[M]$. If $f \in \Sigma_p$ satisfies*

$$|\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z) - 1| < M,$$

then

$$|z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M.$$

Corollary 2.4. *If $\Re\alpha_1 \geq -1/2$ and $f \in \Sigma_p$ satisfies*

$$|z^p H_p^{l,m}[\alpha_1 + 1]f(z) - 1| < M,$$

then

$$|z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M.$$

Proof. This follows from Corollary 2.3 by taking $\phi(u, v, w; z) = v$. ■

Corollary 2.5. *Let $M > 0$ and $0 \neq \alpha_1 \in \mathbb{C}$. If $f \in \Sigma_p$ satisfies*

$$|z^p H_p^{l,m}[\alpha_1 + 1]f(z) - z^p H_p^{l,m}[\alpha_1]f(z)| < \frac{M}{|\alpha_1|}, \text{ then } |z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M.$$

Proof. Let $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = \frac{M}{\alpha_1}z$, $M > 0$. To use Corollary 2.2, we need to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (2.11) is satisfied. This follows since

$$\left| \phi \left(1 + Me^{i\theta}, 1 + \frac{k + \alpha_1}{\alpha_1} Me^{i\theta}, 1 + \frac{L + (2k + \alpha_1)(1 + \alpha_1)Me^{i\theta}}{\alpha_1(1 + \alpha_1)}; z \right) \right| = \frac{kM}{|\alpha_1|} \geq \frac{M}{|\alpha_1|}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\alpha_1 \in \mathbb{C}$ ($\alpha_1 \neq 0, -1$), and $k \geq 1$. The required result now follows from Corollary 2.2.

Theorem 2.4 shows that the result is sharp. The differential equation

$$\frac{zq'(z)}{\alpha_1} = \frac{M}{\alpha_1}z \quad (|\alpha_1| < M)$$

has a univalent solution $q(z) = 1 + Mz$. It follows from Theorem 2.4 that $q(z) = 1 + Mz$ is the best dominant. ■

Next, let us note that

$$\begin{aligned} H_p^{(2,1)}(1, 1; 1)f(z) &= f(z) \\ H_p^{(2,1)}(2, 1; 1)f(z) &= zf'(z) + (1 + p)f(z) \\ H_p^{(2,1)}(3, 1; 1)f(z) &= \frac{1}{2}[z^2 f''(z) + 2(p + 2)zf'(z) + (p + 1)(p + 2)f(z)]. \end{aligned}$$

By taking $l = 2$, $m = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$, Corollary 2.5 shows that for $f \in \Sigma_p$,

$$\text{if } z^p [zf'(z) + pf(z)] < Mz, \text{ then } z^p f(z) < 1 + Mz.$$

Definition 2.3. *Let Ω be a set in \mathbb{C} and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{1}{\alpha_1 + 1} \left(1 + \alpha_1 q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)} \right), (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1, -2, q(\zeta) \neq 0),$$

$$\Re \left\{ \frac{(1 + \alpha_1)v [(\alpha_1 + 2)w - 1 - (1 + \alpha_1)v]}{(1 + \alpha_1)v - (1 + \alpha_1)u} + (1 + \alpha_1)v - (1 + 2\alpha_1u) \right\} \\ \geq k \Re \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in U, \zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Theorem 2.5. Let $\phi \in \Phi_{H,1}[\Omega, q]$. If $f \in \Sigma_p$ satisfies

$$(2.12) \quad \left\{ \phi \left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \right. \right. \\ \left. \left. \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z \right) : z \in U \right\} \subset \Omega,$$

then

$$\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

Proof. Define the analytic function p in U by

$$(2.13) \quad p(z) := \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Then

$$(2.14) \quad \frac{zp'(z)}{p(z)} := \frac{z[H_p^{l,m}[\alpha_1 + 1]f(z)]'}{H_p^{l,m}[\alpha_1 + 1]f(z)} - \frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)}.$$

By making use of (2.3) in (2.14), it follows that

$$(2.15) \quad \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)} = \frac{1}{\alpha_1 + 1} \left(\alpha_1 p(z) + 1 + \frac{zp'(z)}{p(z)} \right).$$

Differentiating logarithmically (2.15), further computations show that

$$(2.16) \quad \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)} = \frac{1}{\alpha_1 + 2} \left(2 + \alpha_1 p(z) + \frac{zp'(z)}{p(z)} \right. \\ \left. + \frac{\alpha_1 zp'(z) + \frac{zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{z^2 p''(z)}{p(z)}}{1 + \alpha_1 p(z) + \frac{zp'(z)}{p(z)}} \right).$$

Define the transformations from \mathbb{C}^3 to \mathbb{C} by

$$(2.17) \quad u = r, \quad v = \frac{1}{\alpha_1 + 1} \left(1 + \alpha_1 r + \frac{s}{r} \right), \\ w = \frac{1}{\alpha_1 + 2} \left(2 + \alpha_1 r + \frac{s}{r} + \frac{\alpha_1 s + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + \frac{t}{r}}{1 + \alpha_1 r + \frac{s}{r}} \right),$$

and let

$$(2.18) \quad \psi(r, s, t; z) := \phi(u, v, w; z) = \phi \left(r, \frac{1}{\alpha_1 + 1} \left[\alpha_1 r + 1 + \frac{s}{r} \right], \frac{1}{\alpha_1 + 2} \left(2 + \alpha_1 r + \frac{s}{r} + \frac{\alpha_1 s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{1 + \alpha_1 r + \frac{s}{r}} \right); z \right).$$

Using the equations (2.13), (2.15) and (2.16), from (2.18), it follows that

$$(2.19) \quad \psi(p(z), zp'(z), z^2p''(z); z) = \phi \left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z \right).$$

Hence (2.12) implies $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. For this purpose, note that

$$\begin{aligned} \frac{s}{r} &= (\alpha_1 + 1)v - (1 + \alpha_1 r), \\ \frac{t}{r} &= (1 + \alpha_1)v[(\alpha_1 + 2)w - 1 - (1 + \alpha_1)v] - \frac{s}{r} \left[(1 + \alpha_1)v - \frac{2s}{r} \right], \end{aligned}$$

and thus

$$\frac{t}{s} + 1 = \frac{(1 + \alpha_1)v [(\alpha_1 + 2)w - 1 - (1 + \alpha_1)v]}{(1 + \alpha_1)v - (1 + \alpha_1 u)} + (1 + \alpha_1)v - (1 + 2\alpha_1 u).$$

Hence $\psi \in \Psi[\Omega, q]$ and by Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z). \quad \blacksquare$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal map $h(z)$ of U onto Ω , the class $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let $\phi \in \Phi_{H,1}[h, q]$ with $q(0) = 1$. If $f \in \Sigma_p$ satisfies*

$$\phi \left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z \right) \prec h(z),$$

then

$$\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

In the particular case $q(z) = 1 + Mz$, $M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$, is simply denoted by $\Phi_{H,1}[\Omega, M]$.

Definition 2.4. Let Ω be a set in \mathbb{C} and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that

$$(2.20) \quad \phi\left(1 + Me^{i\theta}, 1 + \frac{k + \alpha_1(1 + Me^{i\theta})}{(1 + \alpha_1)(1 + Me^{i\theta})}Me^{i\theta}, 1 + \frac{k + \alpha_1(1 + Me^{i\theta})}{(2 + \alpha_1)(1 + Me^{i\theta})}Me^{i\theta} + \frac{(M + e^{-i\theta})[Le^{-i\theta} + kM(1 + \alpha_1 + \alpha_1Me^{i\theta})] - k^2M^2}{(\alpha_1 + 2)(M + e^{-i\theta})[(M + e^{-i\theta})(1 + \alpha_1 + \alpha_1Me^{i\theta}) + kM]}; z\right) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$ and $\Re(Le^{-i\theta}) \geq kM(k - 1)$ for all real θ , $\alpha_1 \in \mathbb{C}$ ($\alpha_1 \neq 0, -1, -2$) and $k \geq 1$.

Corollary 2.6. Let $\phi \in \Phi_{H,1}[\Omega, M]$. If $f \in \Sigma_p$ satisfies

$$\phi\left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is denoted by $\Phi_{H,1}[M]$, and Corollary 2.6 takes the following form:

Corollary 2.7. Let $\phi \in \Phi_{H,1}[M]$. If $f \in \Sigma_p$ satisfies

$$\left|\phi\left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

Corollary 2.8. Let $M > 0$, $\alpha_1 \in \mathbb{C}$ ($\alpha_1 \neq 0, -1$), and $f \in \Sigma_p$ satisfies

$$\left|\frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)} - \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right| < \frac{M^2}{|1 + \alpha_1|(1 + M)},$$

then

$$\left|\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

Proof. This follows from Corollary 2.6 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = \frac{M^2}{(1 + \alpha_1)(1 + M)}z$, $M > 0$. To use Corollary 2.6, we need to show that $\phi \in \Phi_{H,1}[M]$, that is, the admissibility condition (2.20) is satisfied. This follows since

$$\begin{aligned} |\phi(u, v, w; z)| &= \left|1 + \frac{k + \alpha_1(1 + Me^{i\theta})}{(1 + \alpha_1)(1 + Me^{i\theta})}Me^{i\theta} - 1 - Me^{i\theta}\right| \\ &= \frac{M}{|1 + \alpha_1|} \left|\frac{k - 1 - Me^{i\theta}}{1 + Me^{i\theta}}\right| \geq \frac{M}{|1 + \alpha_1|} \left|\frac{k - 1 - M}{1 + M}\right| \end{aligned}$$

$$\geq \frac{M}{|1 + \alpha_1|} \left| \frac{1}{1 + M} - 1 \right| = \frac{M^2}{|1 + \alpha_1|(1 + M)},$$

$z \in U, \theta \in \mathbb{R}, \alpha_1 \in \mathbb{C} (\alpha_1 \neq 0, -1), k \neq 1 + M$ and $k \geq 1$. Hence the result is easily deduced from Corollary 2.6. ■

For $l = 2, m = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$, Corollary 2.8 reduces to the following form:

Example 2.1. If $f \in \Sigma_p$, then

$$\frac{\frac{zf'(z)}{f(z)} \left[\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} - p \right]}{\frac{zf'(z)}{f(z)} + p + 1} \prec p + \frac{M^2}{1 + M}z \Rightarrow \frac{zf'(z)}{f(z)} \prec Mz - p.$$

3. Superordination of the Liu-Srivastava linear operator

The dual problem of differential subordination, that is, differential superordination of the Liu-Srivastava linear operator is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}, q(z) \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), v = \frac{zq'(z) + m\alpha_1q(z)}{m\alpha_1}, \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1)$$

$$\Re \left\{ \frac{(\alpha_1 + 1)(w - u)}{v - u} - (2\alpha_1 + 1) \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U, \zeta \in \partial U$ and $m \geq 1$.

Theorem 3.1. Let $\phi \in \Phi'_H[\Omega, q]$. If $f \in \Sigma_p, z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and

$$\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$$

is univalent in U , then

$$(3.1) \quad \Omega \subset \left\{ \phi \left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z \right) : z \in U \right\}$$

implies

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z).$$

Proof. Let p be defined by (2.2) and ψ by (2.7). Since $\phi \in \Phi'_H[\Omega, q]$, (2.8) and (3.1) yield

$$\Omega \subset \{ \psi(p(z), zp'(z), z^2p''(z); z) : z \in U \}.$$

From (2.6), the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z). \quad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω , then the class $\Phi'_H[h(U), q]$ is written as $\Phi'_H[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $q(z) \in \mathcal{H}$, $h(z)$ be analytic in U and $\phi \in \Phi'_H[h, q]$. If $f \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and $\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$ is univalent in U , then*

$$(3.2) \quad h(z) \prec \phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$$

implies

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z).$$

Theorems 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for an appropriate ϕ .

Theorem 3.3. *Let $h(z)$ be analytic in U , and $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(p(z), \frac{\alpha_1 p(z) + zp'(z)}{\alpha_1}, \frac{z^2 p''(z) + 2(\alpha_1 + 1)zp'(z) + \alpha_1(\alpha_1 + 1)p(z)}{\alpha_1(\alpha_1 + 1)}; z\right) = h(z)$$

has a solution $q \in \mathcal{Q}_1$. If $\phi \in \Phi'_H[h, q]$, $f \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and

$$\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$$

is univalent in U , then

$$h(z) \prec \phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$$

implies

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z),$$

and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.4 and is omitted. ■

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

Corollary 3.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent in U , $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z)$$

is univalent in U , then

$$h_1(z) \prec \phi(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z); z) \prec h_2(z),$$

implies

$$q_1(z) \prec z^p H_p^{l,m}[\alpha_1]f(z) \prec q_2(z).$$

Definition 3.2. Let Ω be a set in \mathbb{C} , and $q(z) \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), v = \frac{1}{\alpha_1 + 1} \left(1 + \alpha_1 q(z) + \frac{zq'(z)}{mq(z)} \right) \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1, -2, q(z) \neq 0)$$

$$\Re \left\{ \frac{(1 + \alpha_1)v[(\alpha_1 + 2)w - 1 - (1 + \alpha_1)v]}{(1 + \alpha_1)v - (1 + \alpha_1)u} + (1 + \alpha_1)v - (1 + 2\alpha_1u) \right\}$$

$$\leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U, \zeta \in \partial U$ and $m \geq 1$.

Now we will give the dual result of Theorem 2.5 for differential superordination.

Theorem 3.4. Let $\phi \in \Phi'_{H,1}[\Omega, q]$. If $f \in \Sigma_p, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ and

$$\phi \left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z \right)$$

is univalent in U , then

$$(3.3) \quad \Omega \subset \left\{ \phi \left(\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_p^{l,m}[\alpha_1 + 3]f(z)}{H_p^{l,m}[\alpha_1 + 2]f(z)}; z \right) : z \in U \right\}$$

implies

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Proof. Let p be defined by (2.13) and ψ by (2.18). Since $\phi \in \Phi'_{H,1}[\Omega, q]$, it follows from (2.19) and (3.3) that

$$\Omega \subset \{ \psi(p(z), zp'(z), z^2p''(z); z) : z \in U \}.$$

From (2.17), the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}. \quad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω , then the class $\Phi'_{H,1}[h(U), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly, the following result is an immediate consequence of Theorem 3.4.

Theorem 3.5. Let $q(z) \in \mathcal{H}$, $h(z)$ be analytic in U and $\phi \in \Phi'_{H,1}[h, q]$. If $f \in \Sigma_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ and $\phi \left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right)$ is univalent in U , then

$$h(z) \prec \phi \left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right)$$

implies

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Theorems 2.6 and 3.5 give the following sandwich-type theorem.

Corollary 3.2. Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent in U , $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, and $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f \in \Sigma_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$, and

$$\phi \left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right)$$

is univalent in U , then

$$h_1(z) \prec \phi \left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q_2(z).$$

Acknowledgement. The authors acknowledge support from the FRGS and Science Fund research grants. This work was completed while the second and third authors were at USM.

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